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Artificial lines are well known to electrical engineers, in telegraphy and telephony, as devices for electrically imitating actual lines of communication in a compact and convenient manner. They are employed industrially in most duplex or quadruplex systems. They are also employed in the laboratory for testing methods of telegraphing, or of telephoning, under conditions that are electrically akin to those of practice.

Artificial telegraph lines contain associated resistance and capacity. Those used in telephony are sometimes provided with inductance and leakance in addition. These quantities are rarely associated distributively, as in actual lines.[1] They are associated for convenience and economy in lumps or sections. Thus an artificial telegraph line containing resistance and capacity AE, Figure 1, may be composed of say four similar sections of resistance AB, BC, CD, and DE, each representing the resistance of say 50 miles (or kilometers) of line. Each section is provided at its centre with a condenser having a capacity of 50 miles of line. The whole line AE will thus purport to represent 200 miles of line. The imitation must, however, be necessarily imperfect, by reason of the lumpiness of the capacity, which is divided into four blocks, and connected to the line at four points only, instead of being distributed uniformly; i. e., indefinitely subdivided, and connected at an infinite number of points, as in the actual line. The smaller the number of sections in the artificial line, the easier and cheaper it will be to build, but the lumpier and more imperfect the imitation will be. The question arises, therefore, as to what are the comparative electrical behaviors of the artificial line and of the line imitated, under any set of assigned conditions.

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1 An exception is found, however, in the artificial lines for duplexing long submarine cables, where the proper proportions of resistance and capacity are associated distributively.

It is the object of this paper to present the quantitative laws that, from the engineering standpoint, control continuous-current artificial lines (sections of resistance and leakance) in the steady state. The basis for the construction of these formulas is given in the Appendix. All of the formulas apply equally to simple alternating-current artificial lines (sections of resistance, inductance, capacity, and leakance) when

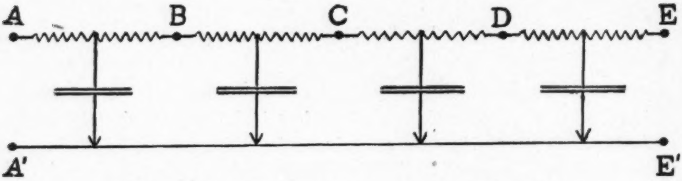


FIGURE 1.— Single-conductor type of artificial line.

interpreted vectorially, or expanded from one dimension to two, in the well-known way; but in order to keep within reasonable limits of space the explicit discussion of alternating-current lines cannot here be considered.

TYPES OF ARTIFICIAL LINE.

There are two types of artificial line; namely, the ground-return-circuit line of Figure 1, and the metallic-return-circuit line of Figure 2,

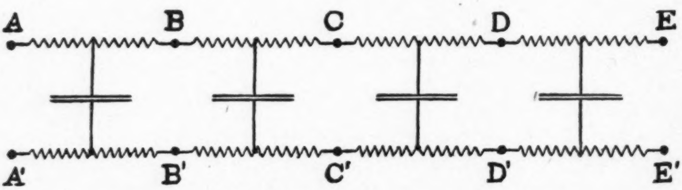


FIGURE 2. — Double-conductor type of artificial line.

which are sometimes respectively defined as the single-conductor and double-conductor artificial lines. The former is characteristic of wire telegraphy, and the latter of wire telephony. In order that two such types of line should be equivalent, ignoring questions of lumpiness, circuit balancing, and circuit symmetry, it is necessary and sufficient that each section AB of Figure 1 should have a resistance $\frac{AB + A'B'}{2}$

of Figure 2, and that the capacity in each section of Figure 1 should be twice the capacity in each section of Figure 2; so that the CR product, i. e. the total resistance R in the line circuit and the total capacity C across the circuit, shall be the same. The double-conductor line of Figure 2 has, therefore, twice the total resistance, and half the total capacity, of the single-conductor line of Figure 1 for the same electrical retardation, and is thus the cheaper type to build, for a given CR. Since, then, to any double-conductor line of Figure 2 there is always a corresponding electrically equivalent single-conductor line of Figure 1, and the latter is, perhaps, the simpler to analyze and discuss, we may confine our attention entirely to the single-conductor or ground-return-circuit artificial line.

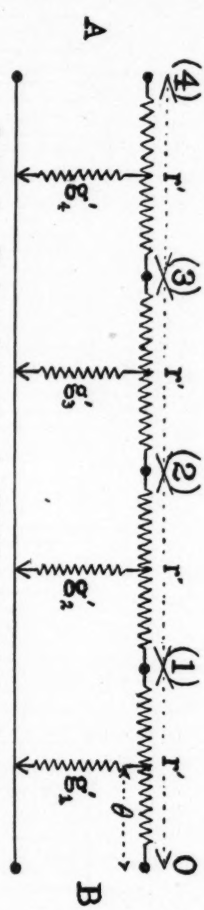


FIGURE 3. — Single-conductor type of artificial line for continuous currents.

FUNDAMENTAL RELATIONS AND NOTATION.

The continuous-current type of single-conductor artificial line is indicated in Figure 3. Let there be m sections. In the case presented, $m = 4$. Let each section represent a nominal length, l, kilometers (or miles) of line, and have a conductor resistance of r' ohms. Let the leak connected to the centre of each section have a conductance of g' mhos and a resistance of $R' = \frac{1}{g'}$ ohms. Let the total nominal length of the line be $L = ml$ kilometers and let $\lambda = \frac{l}{2}$ be the nominal length of a half section in kilometers.

First determine the nominal or apparent attenuation-constant of the artificial line as though the resistance and leakance were distributed as in an actual line:

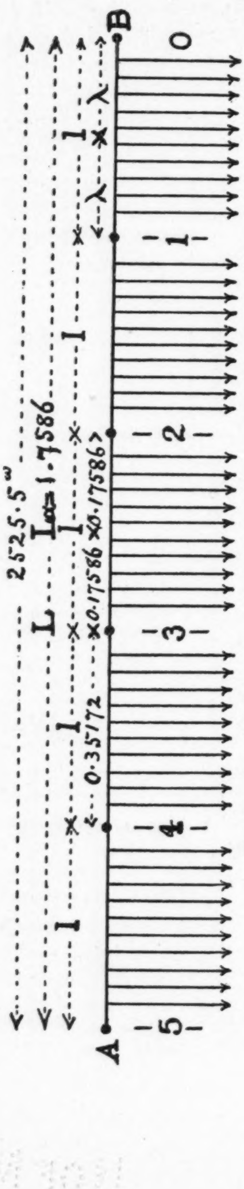


FIGURE 4. — Single-conductor type of real line with distributed leakance.

$$a' = \frac{\sqrt{g'r'}}{l} = \frac{1}{l}\sqrt{\frac{r'}{R'}} \quad \text{hyp. per km.} \quad (1)$$

Call the product la' of the nominal section length l and nominal attenuation-constant the nominal *hyperbolic angle* subtended by the section. Then La' will be the nominal hyperbolic angle of the whole artificial line, and $\lambda a'$ the nominal hyperbolic angle of a half section. These "hyperbolic angles" will be expressed in units of hyperbolic measure corresponding to radians in circular measure, and the unit may be denoted by the abbreviation "hyp."

Find the nominal surge-resistance of the artificial line, as though the resistance and leakance were uniformly distributed.

$$r_o' = \sqrt{\frac{r'}{g'}} = \sqrt{r'R'} \qquad \text{ohms.} \quad (2)$$

The above nominal values of attenuation-constant a', hyperbolic line angles $\lambda a'$, la', and La', as well as the surge-resistance r_o', will then have been obtained as though the resistance r' and leakance g' were presented in an actual uniform line of distributed leakance. They are therefore vitiated by lumpiness. We proceed to correct for lumpiness as follows:

$$\sinh \lambda a = \lambda a' \quad \text{numeric.} \quad (3)$$

that is, the hyperbolic sine of the true semi-section hyperbolic angle is equal to the nominal semi-section hyperbolic angle; or

$$\lambda a = \sinh^{-1}(\lambda a') = \theta \quad \text{hyp.} \quad (4)$$

where θ represents the true semi-sectional hyp.-angle. Similarly, the true value of the surge-resistance, corrected for lumpiness, is

$$r_o = r_o' \cosh \lambda a = r_o' \cosh \theta = r_o' \sqrt{1 + (\lambda a')^2} \quad \text{ohms.} \quad (5)$$

We now obtain from (3), (4) and (5) the true attenuation-constant a of the artificial line, the true surge-resistance r_o, and the true hyperbolic angles λa, la, and La subtended by a half-section, a section, or the whole line, respectively. These various quantities also define the actual line (Figure 4) which the artificial line imitates, after being corrected for lumpiness. The actual line of distributed leakance which is electrically equivalent to an artificial line, after correcting the latter for lumpiness, may be defined as the "*imitated line.*"

As an example, consider an artificial line of $m = 5$ sections, as shown in Figures 5, 6, and 7, with a total nominal length $L = 500$ km.; so that each section has a nominal length $l = 100$ km., and a nominal semi-length $\lambda = 50$ km. The conductor-resistance of each section is $r' = 500$ ohms, corresponding to a nominal linear conductor-resistance of 5 ohms per km. The leak of each section has a resistance $R' = 4000$ ohms or a conductance of 0.00025 mho (0.25 millimho), corresponding to a nominal linear leakance of 2.5 micromhos per km., or a linear insulation-resistance of 400,000 km.-ohms. The nominal attenuation-constant of the artificial line will be, by (1), $a' = 0.0035355$ hyp. per km. The nominal hyperbolic angle subtended by a half-section, a section, and the whole line, will be respectively $\theta = \lambda a' = 0.17678$, $la' = 0.35355$, $La' = 1.7678$ hyps. The nominal surge-resistance will be by (2) $r_o' = 1414.2$ ohms. We must now find the corrected values for these quantities corresponding to the imitated line shown in Figure 4.

With reference to formula (3), we find in tables of hyperbolic functions [2] that the angle whose hyperbolic sine is 0.17678 must be $\lambda a = 0.17586$ hyp.; which is the true angle of a semi-section of the artificial line, corrected for lumpiness. The true angle subtended by a section will be $la = 0.35172$ hyp., and by the whole line 1.7586 hyps. The true attenuation-constant of the artificial line, or the natural attenuation-constant of the imitated line, will be $a = 0.0035172$ hyp. per km. The true surge-resistance by (5) $r_o = 1436.13$ ohms. In other words the artificial line will behave externally in all respects, after the steady state has been attained, as though it were an actual smooth line of distributed leakance with these corrected constants. The correction

[2] The best tables probably are "Tafeln der Hyperbelfunctionen und der Kreisfunctionen" by Dr. W. Ligowski, Berlin, Ernst & Korn, 1890.

has in this case diminished the nominal attenuation-constant and hyperbolic line angles by 0.52 per cent, but has increased the surge-resistance by 1.55 per cent. The linear conductor-resistance of the imitated line, Figure 4, will be $ar_o = 5.051$ ohms per km. The linear leakance of the imitated line will be $a/r_o = 2.44989 \times 10^{-6}$ mho per km., corresponding to a linear insulation resistance of 408,320 km.-ohms.

Figures 5, 6, and 7 are diagrams of the voltage and current distribution over the artificial line above defined, for the respective cases of line grounded, freed, and grounded through 750 ohms, at B, the distant end. The steady impressed emf. at the sending end A is assumed as 100 volts in each case. Conductances are written in millimhos. All of the numerical work on these diagrams was carried out by the ordinary formulas of Ohm's law, and inspection will show that the arithmetical results are consistent. The various formulas given in this paper admit, therefore, of being checked by reference to these diagrams.

Artificial Line freed at Far End. (Figure 6.)

Sending-End Resistance.

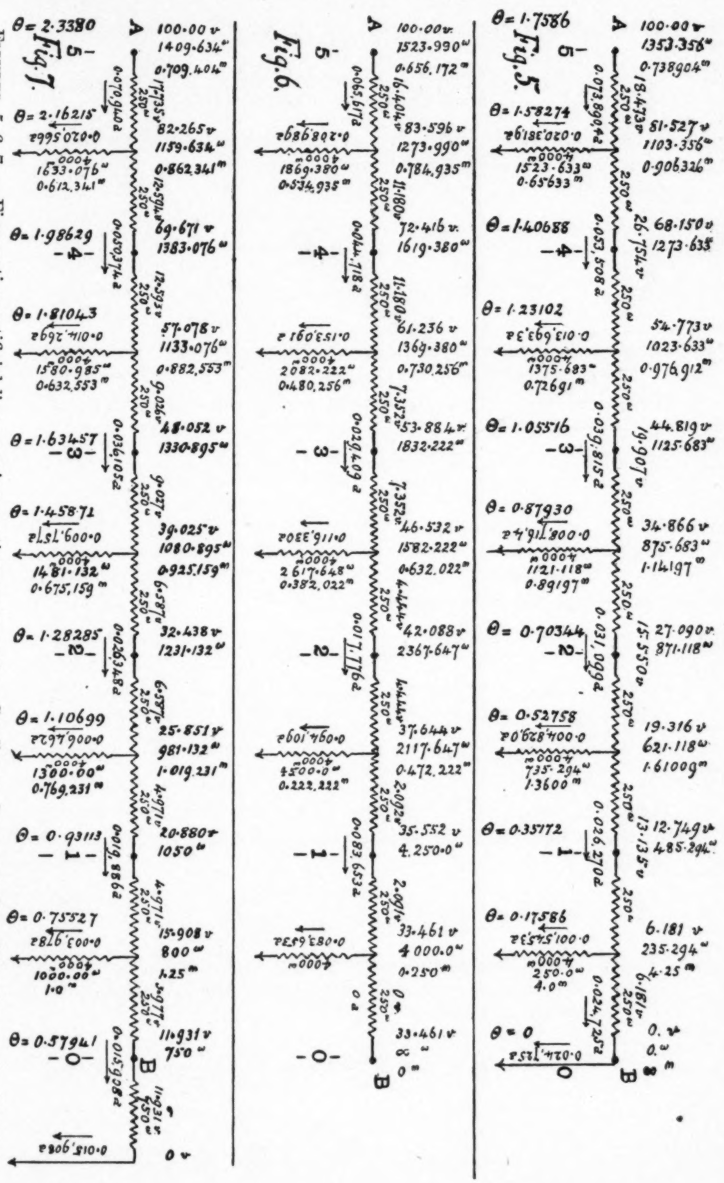
The sending-end resistance of an artificial line at the nth junction; i. e., the resistance offered to ground by the line, at and beyond the nth junction, is

$$R_f = r_o \coth L_2 a = r_o \coth 2\,n\theta \qquad \text{ohms,} \quad (6)$$

where L_2 is the length of the line in km. reckoned from the far free end. When the sending-end resistance is measured at A, Figure 6, so as to include the whole line, $L_2 = L$, and $n = m$. As L_2 increases from 0 to ∞, coth $L_2 a$ diminishes from ∞ to 1. Thus, in Figure 6, with $r_o =$ 1436.1 ohms, and $m = 5$; or $L = 500$, $L_2 a = 1.7586 = 2\,m\theta$, coth $2\,m\theta = 1.0612$, and $R_f = 1436.1 \times 1.0612 = 1523.99$ ohms, as indicated at A. In the case of a smooth actual line, such as is shown in Figure 4, L_2 may be varied continuously between 0 and L kms.; but in an artificial line, L_2 can only be varied in steps of $2\,\theta$. That is, formula (6) applies to all points of the imitated line, but only to the junction points of the artificial line.

At the nth leak, excluding the same, the sending-end resistance is

$$R'_{n,f} = r_o' \frac{\cosh (2\,n - 1)\,\theta}{\sinh (2\,n - 2)\,\theta} = \frac{r_o \cosh (2\,n - 1)\,\theta}{\cosh \theta \sinh (2\,n - 2)\,\theta} \qquad \text{ohms.} \quad (7)$$



FIGURES 5, 6, 7. — Five-section artificial line carrying continuous currents. Sending end connected to 100 volts e.m.f. Distant end grounded, freed, and to ground through resistance. In Figure 6 the indicated leak currents should be divided by ten.

At the nth leak, including the same, the sending-end resistance is

$$R'_{f,n} = r_o' \frac{\cosh (2n - 1)\theta}{\sinh 2n\theta} = \frac{r_o \cosh (2n - 1)\theta}{\cosh \theta \sinh 2n\theta} \quad \text{ohms.} \quad (8)$$

When the number of sections of artificial line becomes indefinitely great, the two immediately preceding expressions respectively become:

$$R'_{\infty,f} = r_o' \epsilon^{\theta} = \frac{r_o \epsilon^{\theta}}{\cosh \theta} \quad \text{ohms,}$$

and

$$R'_{f,\infty} = r_o' \epsilon^{-\theta} = \frac{r_o \epsilon^{-\theta}}{\cosh \theta} \quad \text{ohms,} \quad (9)$$

where ϵ is the base of Napierian logarithms.

The ratio of the sending-end resistance at and excluding the $(n + 1)$th leak to that at and including the nth leak is

$$\frac{R'_{n+1,f}}{R'_{f,n}} = \frac{\cosh (2n + 1)\theta}{\cosh (2n - 1)\theta}. \quad (10)$$

This is the ratio of the extreme sending-end resistances, when ascending from one leak where it is a local minimum, to the next higher leak where it is a local maximum. When the artificial line becomes indefinitely long, this ratio tends to the limit $\epsilon^{2\theta}$.

Voltage. Far End Free.

The voltage e_o at the far free end of the artificial line, Figure 6, will be:

$$e_o = \frac{e_m}{\cosh 2m\theta} = \frac{e_m}{\cosh L_2 a} \quad \text{volts,} \quad (11)$$

where e_m is the voltage impressed on the mth junction, or sending end. If the voltage ϵ_n should be impressed on the line at the nth leak, the formula is

$$e_o = \frac{\epsilon_n \cosh \theta}{\cosh (2n - 1)\theta} \quad \text{volts.} \quad (12)$$

Thus, if $e_m = 100$ volts, and $m = 5$, as in Figure 6, $2m\theta = 1.7586$ hyps. and $\cosh 2m\theta = 2.9883$; so that $e_o = 100/2.9883 = 33.46$ volts.

The voltage at junction (n) is

$$e_n = e_o \cosh 2n\theta = e_m \frac{\cosh 2n\theta}{\cosh 2m\theta} \quad \text{volts.} \quad (13)$$

The voltage at the nth leak is

$$\epsilon_n = e_o \frac{\cosh (2n-1)\theta}{\cosh \theta} = \epsilon_m \frac{\cosh (2n-1)\theta}{\cosh (2m-1)\theta} \quad \text{volts.} \quad (14)$$

Consequently, the voltages at successive junctions, $e_o, e_1, e_2, \ldots e_n$, are respectively proportional to $\cosh 0$, $\cosh 2\theta$, $\cosh 4\theta, \ldots \cosh 2n\theta$; that is, to the cosine of the hyperbolic angle of the junction, measured from the far free end.

Similarly, the voltages at successive leaks, $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ are respectively proportional to $\cosh\theta$, $\cosh 3\theta, \ldots \cosh (2n-1)\theta$; that is, to the hyperbolic angle of the leak, measured from the far free end.

As we ascend along the line by steps of θ from the far free end, the voltages increase as follows:

Angular Distance from far free end. Hyps.	Point.	Voltage Symbol.	Value. Volts.
0	End	e_o	e_o
θ	Leak 1	ϵ_1	$e_o \frac{\cosh\theta}{\cosh\theta}$
2θ	Junction 1	e_1	$e_o \cosh 2\theta$
3θ	Leak 2	ϵ_2	$e_o \frac{\cosh 3\theta}{\cosh\theta}$
4θ	Junction 2	e_2	$e_o \cosh 4\theta$
. . .	. . . . . .	. . .	. . . . . .
$(2n-1)\theta$	Leak n	ϵ_n	$e_o \frac{\cosh (2n-1)\theta}{\cosh\theta}$
$2n\theta$	Junction n	e_n	$e_o \cosh 2n\theta$

Current Strength. Far End Free.

The current strength at the sending end is:

$$I_m = \frac{e_m}{r_o \coth L_2 a} = \frac{e_m}{r_o \coth 2m\theta} \quad \text{amperes,} \quad (15)$$

where e_m is the voltage impressed on the mth junction.

The current strength at the nth junction is:

$$I_n = I_m \frac{\sinh 2n\theta}{\sinh 2m\theta} \quad \text{amperes.} \quad (16)$$

Thus in Figure 6 the current at the sending end is 0.065617 ampere. The current at junction 3 will be $0.065617 \times \dfrac{\sinh 1.05516}{\sinh 1.7586} = 0.029409$ ampere.

At the nth leak, the ratio of ongoing to arriving line current is

$$\frac{I_{n-1}}{I_n} = \frac{\sinh 2\,(n-1)\theta}{\sinh 2\,n\theta}. \tag{17}$$

The current escaping at the nth leak is:

$$i_n = \epsilon_n g' = e_o g' \frac{\cosh (2\,n-1)\,\theta}{\cosh \theta} = \epsilon_m g' \frac{\cosh (2\,n-1)\,\theta}{\cosh (2\,m-1)\,\theta}$$
$$= e_m g' \frac{\cosh (2\,n-1)\,\theta}{\cosh \theta \cosh 2\,m\theta} \qquad \text{amperes.} \tag{18}$$

LINE GROUNDED AT FAR END. (Figure 5.)

Sending-End Resistance.

The sending-end resistance at the nth junction with the far end grounded is:

$$R_g = r_o \tanh L_2 a = r_o \tanh 2\,n\theta \qquad \text{ohms.} \tag{19}$$

In the case represented by Figure 5, for $m = 5$, $R_g = 1436.1 \times 0.94235 = 1353.3$ ohms. As we ascend the line from junction to junction, the resistances are in proportion to the hyp. tangents of the angles of those junctions.

The sending-end resistance at and excluding the nth leak is:

$$R'_{n,g} = r_o' \frac{\sinh (2\,n-1)\,\theta}{\cosh (2\,n-2)\,\theta} \qquad \text{ohms.} \tag{20}$$

The sending-end resistance at and including the nth leak is:

$$R'_{g,n} = r_o' \frac{\sinh (2\,n-1)\,\theta}{\cosh 2\,n\theta} \qquad \text{ohms.} \tag{21}$$

When n is indefinitely increased, (20) becomes:

$$R'_{\infty,g} = r_o' \epsilon^{\theta} \qquad \text{ohms,} \tag{22}$$

and (21) becomes:

$$R'_{g,\infty} = r_o' \epsilon^{-\theta} \qquad \text{ohms.} \tag{23}$$

The ratio of local maximum resistance just before a leak to the local minimum resistance just after the preceding leak is:

$$\frac{R'_{n+1,g}}{R'_{g,n}} = \frac{\sinh(2n+1)\theta}{\sinh(2n-1)\theta}. \tag{24}$$

When n is increased indefinitely, this ratio becomes:

$$\frac{R'_{\alpha,g}}{R'_{g,\alpha}} = \epsilon^{2\theta}. \tag{25}$$

Receiving-End Resistance. Far End grounded.

The receiving-end resistance, or resistance which the artificial line appears to offer, as judged by an observer at the far end, from the received current to ground and the impressed emf. at the sending end, is:

$$R_l = r_o \sinh L_2\alpha = r_o \sinh 2m\theta \qquad \text{ohms.} \tag{26}$$

In the case of Figure 5, $R_l = 1436.1 \sinh 1.7586 = 1436.1 \times 2.81602 = 4044.2$ ohms. The received current to ground at the far end will therefore be $100/4044.2 = 0.02472$ ampere.

Voltage. Far End grounded.

The emf. at the nth junction in terms of the emf. e_m impressed on the mth junction is:

$$e_n = e_m \frac{\sinh 2n\theta}{\sinh 2m\theta} \qquad \text{volts,} \tag{27}$$

or, in terms of the current i_o to ground at the far end, it is;

$$e_n = I_o r_o \sinh 2n\theta \qquad \text{volts.} \tag{28}$$

Consequently, the voltages at successive ascending junctions are proportional to the hyperbolic sines of the angles of those junctions.

The emf. at the nth leak is:

$$\epsilon_n = \epsilon_m \frac{\sinh(2n-1)\theta}{\sinh(2m-1)\theta} \qquad \text{volts,} \tag{29}$$

in terms of the emf. ϵ_m at the mth leak; or

$$\epsilon_n = I_o r_o \frac{\sinh(2n-1)\theta}{\cosh\theta} = I_o r_o' \sinh(2n-1)\theta \qquad \text{volts,} \tag{30}$$

in terms of the current to ground and of the surge-resistances, corrected and nominal.

Consequently, the voltages at successive ascending leaks are proportional to the hyperbolic sines of the angles of those leaks.

Current. Far End grounded.

The current at the sending end is:

$$I_m = \frac{e_m}{r_o \tanh L_2 a} = \frac{e_m}{r_o \tanh 2\,m\theta} \quad \text{amperes.} \quad (31)$$

The current at junction n is:

$$I_n = I_m \frac{\cosh 2\,n\theta}{\cosh 2\,m\theta} = \frac{e_m}{r_o} \cdot \frac{\cosh 2\,n\theta}{\sinh 2\,m\theta} \quad \text{amperes.} \quad (32)$$

The current at junction 0, or the grounded end, is:

$$I_o = \frac{e_m}{r_o \sinh 2\,m\theta} = \frac{e_m}{r_o \sinh L_2 a} = \frac{I_m}{\cosh 2\,m\theta} = \frac{I_m}{\cosh L_2 a} \quad \text{amperes.} \quad (33)$$

At the nth leak, the ratio of ongoing to arriving line current is:

$$\frac{I_{n-1}}{I_n} = \frac{\cosh 2\,(n-1)\,\theta}{\cosh 2\,n\theta}. \quad (34)$$

The current escaping at the nth leak is:

$$i_n = \epsilon_n g' = \epsilon_m g' \frac{\sinh (2\,n-1)\,\theta}{\sinh (2\,m-1)\,\theta} = 2\,I_o \sinh \theta \sinh (2\,n-1)\theta \quad \text{amperes.} \quad (35)$$

By comparing formulas (6) and (19), (13) and (27), (16) and (32), it will be seen that with the far end free, the sending-end resistances follow the cotangents, voltages the cosines, and currents the sines, of the hyp. angles of the junctions; but that with the far end grounded, the sending-end resistances follow the tangents, voltages the sines, and currents the cosines, of said angles.

Line grounded at Far End through a Resistance σ. (Figure 7.)

First Case. Let σ be not greater than r_o.

Find the hyperbolic angle ϕ of the terminal load σ from

$$\tanh \phi = \frac{\sigma}{r_o}. \quad (36)$$

Then treat the artificial line as grounded directly, but with the angles of all its leaks and junctions increased by ϕ. Formulas (19) to (35) will then apply, except where the strength of the current to ground enters into consideration, as in (26), (28), (30), and (33). The surge-resistance r_o must then be replaced by a new surge-resistance

$$r_o'' = \frac{r_o}{\cosh \phi} \quad \text{ohms.} \quad (37)$$

Thus, the sending-end resistance becomes, by (19):

$$R_{g\sigma} = r_o \tanh (L_2 a + \phi) = r_o \tanh (2\,m\theta + \phi) \quad \text{ohms.} \quad (38)$$

The resistance at and excluding the nth leak becomes, by (20):

$$R'_{n,g\sigma} = r_o' \frac{\sinh [(2\,n - 1)\,\theta + \phi]}{\cosh [(2\,n - 2)\,\theta + \phi]} \quad \text{ohms.} \quad (39)$$

The resistance at and including the nth leak becomes, by (21):

$$R'_{g,n\sigma} = r_o' \frac{\sinh [(2\,n - 1)\,\theta + \phi]}{\cosh (2\,n\theta + \phi)} \quad \text{ohms.} \quad (40)$$

The ratio of local maximum resistance just before a leak to the local minimum just after the preceding leak is:

$$\frac{R'_{n+1,g\sigma}}{R'_{g,n\sigma}} = \frac{\sinh [(2\,n + 1)\,\theta + \phi]}{\sinh [(2\,n - 1)\,\theta + \phi]}. \quad (41)$$

For example, the sending-end resistance of the line in Figure 7, with $\sigma = 750$ ohms, whose hyperbolic angle is $\tanh^{-1} \frac{750}{1436.1} = 0.57941$ hyp., becomes by (38), $1436.1 \times \tanh 2.338 = 1409.6$ ohms.

The receiving-end resistance is, by (26) and (37):

$$\begin{aligned} R_{l\sigma} &= r_o'' \sinh (L_2 a + \phi) = r_o'' \sinh (2\,m\theta + \phi) \\ &= r_o \sinh 2\,m\theta + \sigma \cosh 2\,m\theta \end{aligned} \quad \text{ohms.} \quad (42)$$

Thus, in Figure 7, $r_o'' = 1224.7$ ohms by (37), and

$$R_{l\sigma} = 1224.7 \sinh 2.338 = 6285.4 \text{ ohms.}$$

The voltage at the nth junction is, by (27) and (28):

$$e_n = e_m \frac{\sinh (2\,n\theta + \phi)}{\sinh (2\,m\theta + \phi)} = I_o r_o'' \sinh (2\,n\theta + \phi) \quad \text{volts;} \quad (43)$$

so that the voltage at the distant end of the line is:

$$e_o = e_m \frac{\sinh \phi}{\sinh (2\,m\theta + \phi)} = I_o r_o'' \sinh \phi \qquad \text{volts.} \qquad (44)$$

Thus, in Figure 7, the voltage at the distant end B is

$$100 \times \frac{\sinh 0.5794}{\sinh 2.338} = 11.931.$$

The voltage at the nth leak is, by (29) and (30):

$$\epsilon_n = \epsilon_m \frac{\sinh [(2\,n - 1)\,\theta + \phi]}{\sinh [(2\,m - 1)\,\theta + \phi]} = I_o r_o'' \frac{\sinh [(2\,n - 1)\theta + \phi]}{\cosh \theta} \qquad \text{volts.} \qquad (45)$$

The current at the sending end is, by (31):

$$I_{m\sigma} = \frac{e_m}{r_o \tanh (L_2\alpha + \phi)} = \frac{e_m}{r_o \tanh (2\,m\theta + \phi)} \qquad \text{amperes.} \qquad (46)$$

At junction n it is, by (32):

$$I_{n\sigma} = I_{m\sigma} \frac{\cosh (2\,n\theta + \phi)}{\cosh (2\,m\theta + \phi)} = \frac{e_m}{r_o} \frac{\cosh (2\,n\theta + \phi)}{\sinh (2\,m\theta + \phi)} \qquad \text{amperes.} \qquad (47)$$

At the distant end, through σ, it is, by (33):

$$I_{o\sigma} = \frac{e_m}{r_o'' \sinh (2\,m\theta + \phi)} = \frac{e_m}{r_o'' \sinh (L_2\alpha + \phi)}$$

$$= \frac{I_{m\sigma} \cosh \phi}{\cosh (2\,m\theta + \phi)} = \frac{I_{m\sigma} \cosh \phi}{\cosh (L_2\alpha + \phi)} \qquad \text{amperes.} \qquad (48)$$

At the nth leak, the ratio of ongoing to arriving current is, by (34):

$$\frac{I_{\sigma,\,n-1}}{I_{\sigma,\,n}} = \frac{\cosh [2\,(n - 1)\,\theta + \phi]}{\cosh (2\,n\theta + \phi)}. \qquad (49)$$

For example, the received current to ground through σ is, by (48), 100/6285.4 = 0.01591 ampere.

Second Case, with σ not less than r_o.

Find the hyperbolic angle of the terminal load σ from the formula:

$$\tanh \phi' = \frac{r_o}{\sigma}. \qquad (50)$$

Then treat the artificial line, actually grounded through σ, as though it were freed at the far end, but with its angular length increased at all

points by ϕ' hyps. Formulas (6) to (18) will then apply, except that where the strength of the received current to ground enters into consideration, as in (56) and (62), the surge-resistance r_o must be replaced by a new surge-resistance:

$$r_o''' = \frac{r_o}{\sinh \phi'} \qquad \text{ohms.} \quad (51)$$

Thus, the sending-end resistance at junction n becomes, by (6):

$$R_{f\sigma} = r_o \coth (L_2 a + \phi') = r_o \coth (2\,n\theta + \phi') \quad \text{ohms.} \quad (52)$$

The resistance at the nth leak, excluding the same, is, by (7):

$$R'_{n,f\sigma} = r_o' \frac{\cosh [(2\,n - 1)\,\theta + \phi']}{\sinh [(2\,n - 2)\,\theta + \phi']} \qquad \text{ohms.} \quad (53)$$

The resistance at the nth leak, including the same, is, by (8):

$$R'_{f\sigma,n} = r_o' \frac{\cosh [(2\,n - 1)\,\theta + \phi']}{\sinh (2\,n\theta + \phi')} \qquad \text{ohms.} \quad (54)$$

The ratio of resistance at and excluding the $(n + 1)$th leak to that at and including the nth leak is, by (10):

$$\frac{R'_{n+1,g\sigma}}{R'_{g\sigma,n}} = \frac{\cosh [(2\,n + 1)\,\theta + \phi']}{\cosh [(2\,n - 1)\,\theta + \phi']}. \qquad (55)$$

The receiving-end resistance is, by (26):

$$\begin{aligned} R_{l\sigma} &= r_o''' \cosh (2\,m\theta + \phi') = r_o''' \cosh (L_2 a + \phi') \\ &= r_o \sinh 2\,m\theta + \sigma \cosh 2\,m\theta \qquad \text{ohms.} \quad (56) \end{aligned}$$

The voltage at junction n is, by (13):

$$e_{n\sigma} = e_m \frac{\cosh (2\,n\theta + \phi')}{\cosh (2\,m\theta + \phi')} = e_o \frac{\cosh (2\,n\theta + \phi')}{\cosh \phi'} \qquad \text{volts.} \quad (57)$$

At the distant end, or junction 0, it is:

$$e_{o\sigma} = \frac{e_m \cosh \phi'}{\cosh (2\,m\theta + \phi')} = \frac{e_m \cosh \phi'}{\cosh (L_2 a + \phi')} \qquad \text{volts.} \quad (58)$$

At the nth leak, it is, by (14):

$$\epsilon_{n\sigma} = \epsilon_m \frac{\cosh [(2\,n - 1)\,\theta + \phi']}{\cosh [(2\,m - 1)\,\theta + \phi']} = \frac{e_o \cosh [(2n - 1)\theta + \phi']}{\cosh \theta \cosh \phi'} \text{ volts.} \quad (59)$$

The current strength at the sending end or junction m is:

$$I_{m\sigma} = \frac{e_m}{r_o \coth (L_2 a + \phi')} = \frac{e_m}{r_o \coth (2\,m\theta + \phi')} \quad \text{amperes.} \qquad (60)$$

At junction n it is, by (16):

$$I_{n\sigma} = I_{m\sigma} \frac{\sinh (2\,n\theta + \phi')}{\sinh (2\,m\theta + \phi')} \quad \text{amperes.} \qquad (61)$$

At the receiving end, or junction 0, it is:

$$I_{o\sigma} = I_{m\sigma} \frac{\sinh \phi'}{\sinh (2\,m\theta + \phi')} = \frac{e_m}{r_o''' \cosh (2\,m\theta + \phi')} \quad \text{amperes.} \qquad (62)$$

At the nth leak the ratio of ongoing to arriving line current is, by (17):

$$\frac{I_{n-1,\sigma}}{I_{n\sigma}} = \frac{\sinh [2\,(n-1)\,\theta + \phi']}{\sinh (2\,n\theta + \phi')}. \qquad (63)$$

The current escaping at the nth leak is, by (18):

$$i_{n\sigma} = \epsilon_{n\sigma} g' = e_{o\sigma} g' \frac{\cosh [(2\,n-1)\theta + \phi']}{\cosh (\theta + \phi')} = \epsilon_{m\sigma} g' \frac{\cosh [(2\,n-1)\theta + \phi']}{\cosh [(2\,m-1)\theta + \phi']}$$

$$= e_{m\sigma} g' \frac{\cosh [(2\,n-1)\,\theta + \phi']}{\cosh \theta \cosh (2\,m\theta + \phi')} \quad \text{amperes.} \qquad (64)$$

As an example, let $\sigma = 3750$ ohms. Then $\phi' = \tanh^{-1} \frac{1436.13}{3750} =$ 0.403535 hyp. The sending-end resistance at junction 1 is, by (52), 1436.1 coth 0.755255 = 2250 ohms, which by Figure 8 is evidently correct. Again, the received current strength for the same case with $e_1 = 10$ volts, at junction 1, will be,by (62), $I_{o\sigma} = \frac{10}{3464.1 \cosh 0.75526}$ = 0.00222 amperes, which is also easily seen to be correct, from Figure 8.

Third Case $\sigma = r_o$. Exponential Case.

In the particular and intermediate case in which $\sigma = r_o$, either of the preceding sets of formulas applies under limit conditions. We have $\phi = \phi' = \infty$, by (36) and (50). Consequently, the sending-end resistance becomes at any junction:

$$R_{\sigma r_o} = r_o \quad \text{ohms.} \qquad (65)$$

The resistance at any leak, excluding the same, is:

$$R'_{\sigma r_o} = r_o' \,\epsilon^{\theta} \quad \text{ohms.} \qquad (66)$$

ϵ being the Napierian base.

The resistance at any leak, including the same, is:

$$R'_{r_0 g} = r_o' \epsilon^{-\theta} \quad \text{ohms.} \quad (67)$$

Thus, in Figure 9, where one section of artificial line is grounded at the distant end through a resistance $\sigma = r_o = 1436.13$ ohms, the

Fig. 8

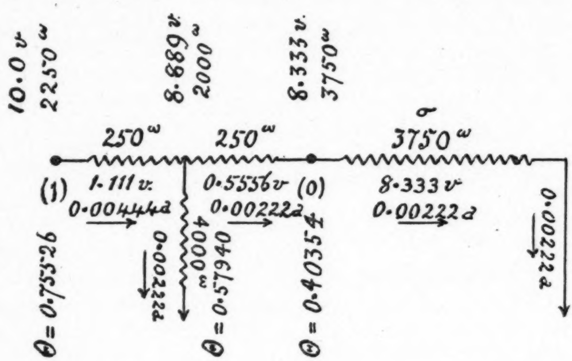
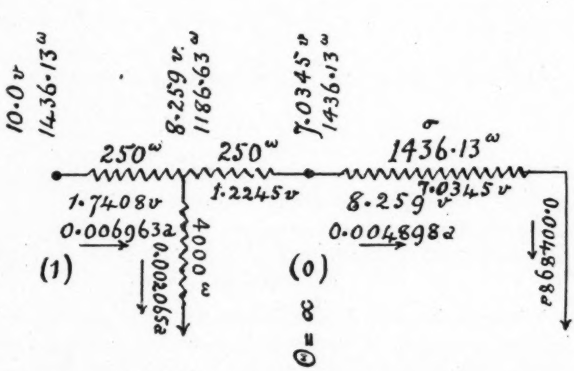


Fig. 9



sepuing-end resistance at junction (1) is r_o, at leak 1 is $1414.2 \times \epsilon^{0.17586} = 1686.1$ ohms, excluding the leak, and $1414.2\ \epsilon^{-0.17586} = 1186.13$ ohms, including the leak.

The receiving end resistance is:

$$R_{lr_0} = r_o\, \epsilon^{L_2 a} = r_o\, \epsilon^{2m\theta} \quad \text{ohms.} \quad (68)$$

Thus in Figure 9 the receiving-end resistance is $1436.1 \times 1.422 =$ 2042 ohms.

The voltage at junction n is:

$$e_{nr_0} = e_m \epsilon^{2\theta(n-m)} = e_o \epsilon^{2n\theta} \qquad \text{volts.} \qquad (69)$$

At the distant end, or junction 0, it is:

$$e_{or_0} = e_m \epsilon^{-L_2 a} = e_m \epsilon^{-2m\theta} \qquad \text{volts.} \qquad (70)$$

At the nth leak, it is:

$$\epsilon_{nr_0} = \epsilon_m \epsilon^{2\theta(n-m)} = e_o \frac{\epsilon^{(2n-1)\theta}}{\cosh \theta} \qquad \text{volts.} \qquad (71)$$

The current strength at the sending end, or junction m, is:

$$I_{mr_0} = \frac{e_m}{r_o} \qquad \text{amperes.} \qquad (72)$$

At junction n it becomes:

$$I_{nr_0} = I_m \epsilon^{2\theta(n-m)} = I_o \epsilon^{2n\theta} \qquad \text{amperes.} \qquad (73)$$

At the receiving end it is:

$$I_{or_0} = \frac{e_m}{r_o} \epsilon^{-L_2 a} = \frac{e_m}{r_o} \epsilon^{-2m\theta} = I_m \epsilon^{-2m\theta} \qquad \text{amperes.} \qquad (74)$$

At any leak the ratio of ongoing to arriving line current is

$$\frac{I_{n-1, r_0}}{I_{n r_0}} = \epsilon^{-2\theta}. \qquad (75)$$

General Propositions.

Equal Increase of Receiving-End Resistance due to Resistance inserted at either End of Line.

When a resistance σ is added to the line at the sending end, the sending-end resistance is obviously increased by σ; but the receiving-end resistance is increased by $\sigma \cosh La = \sigma \cosh 2\, m\theta$ ohms. Comparing this result with formula (42), it is evident that a resistance σ adds $\sigma \cosh 2\, m\theta$ to the resistance of an artificial line, or $\sigma \cosh La$ to that of a smooth line, whether it be added at the sending or receiving end. Thus, if to the sending end of the artificial line in Figure 5, a resistance of 750 ohms be added, the sending-end resistance will be increased to 2103.4 ohms, and the voltage at the end A of the artificial line will

thereby be reduced from 100 to 64.343 volts. The effect of this will be to reduce the received current at B to 0.01591 ampere, the same as in Figure 7.

Best Resistance of Receiving Instrument.

Electromagnetic receiving instruments may be divided into two classes; viz. (1) those, as of the movable-coil type, in which the magneto-mechanical force, or torque, is directly proportional to the ampere-turns, and (2) those, like simple non-polarized relays, in which the magneto-mechanical force, or torque, may be nearly proportional to the square of the ampere-turns at low magnetic saturation, but, as saturation increases, to perhaps a lower power of ampere turns than the first. In either case, the magneto-mechanical force may be expressed by:

$$F = a\,(I_o n_1)^p \quad \text{dynes or dyne-perp. cms.,} \qquad (76)$$

where F is the force or torque, a is a constant of the instrument, I_o is the received current in amperes, n_1 the number of turns in the winding, and p some exponent not greater than 2. The received current I_o is expressed by (42) or (56). The number of turns n_1 in a given winding space is well known to be sensibly proportional to $\sqrt{\sigma}$, where σ is the resistance of the winding in ohms, provided that the size of copper wire selected is within the fairly wide range that keeps the ratio of covered diameter to bare diameter sensibly constant. Consequently, we have approximately:

$$F = a'\left(\frac{e_m\sqrt{\sigma}}{r_o \sinh 2\,m\theta + \sigma \cosh 2\,m\theta}\right)^p \quad \text{dynes, or dyne-perp. cms.} \qquad (77)$$

In order to make this force a maximum by varying σ, we differentiate F with respect to σ in the usual way, and equate to zero. We then obtain

$$\sigma = r_o \tanh 2\,m\theta = r_o \tanh La \qquad \text{ohms.} \qquad (78)$$

That is, the best resistance for the electromagnetic winding of the receiver is equal to the sending-end resistance R_g of the line, no matter what the exponent p which expresses the relation between torque and ampere-turns.[3]

[3] See Ayrton and Whitehead paper in Bibliography.

Imitative Accuracy of Artificial Lines.

As the preceding formulas indicate, an artificial line does not correspond electrically to the real smooth line having the same linear constants (resistance and leakance per km.) as its nominal linear constants, but to some other real smooth line having somewhat different linear constants. In other words, an artificial line has an imitation error due to lumpiness. The amount of this error will differ with the degree of lumpiness, and would obviously disappear if the number of line sections were made indefinitely great. In general, the fewer the sections the greater the lumpiness, and the greater the lumpiness error. With any given artificial line, however, the lumpiness error depends upon the particular quantity considered, and is not the same for all quantities. Thus, let a' and r_o' be the nominal attenuation-constant and surge-resistance of the uncorrected artificial line, by (1) (2); while a and r_o are the corresponding constants, corrected for lumpiness, according to (4) and (5). Then the ratio of received ground current over the artificial line to that over the real line of same nominal linear constants will be $\frac{r_o' \sinh La'}{r_o \sinh La}$. Again, the ratio of sending-end resistances with the far end grounded will be $\frac{r_o \tanh La}{r_o' \tanh La'}$, a distinctly different ratio; while in respect to, say, voltage at the free distant end, the ratio will be again different. Consequently there is no single correction factor for the lumpiness of an artificial line, and each particular quantity will have to be corrected, according to the preceding formulas.

Equivalence between Single-Section Artificial Lines and Uniform Smooth Lines.

In Figure 10, let AOB represent a uniform smooth actual line of L kms. in length, with a linear conductor-resistance of r ohms per km. and a linear dielectric conductance of g mhos per km. Its attenuation-constant will then be $a = \sqrt{gr}$ hyps. per km., and its surge-resistance $r_o = \sqrt{\frac{r}{g}}$ ohms. Its hyperbolic angle will be La, and its semi-hyperbolic angle λa hyps. Then let a single section of artificial line be constructed, as in Figure 11, with a total conductor-resistance of r' ohms, a leak at the centre of g' mhos, or $R' = 1/g'$ ohms. This single section of artificial line will be the complete external equivalent of the actual uniform line in Figure 10 in the steady state, if:

Fig. 10

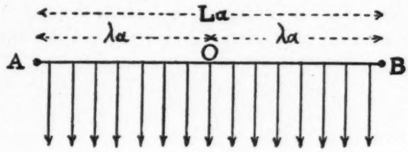


Fig. 11

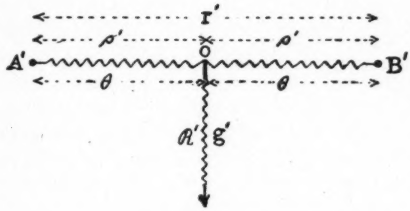
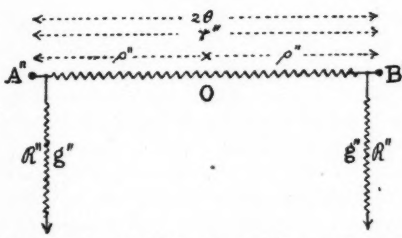


Fig. 12



FIGURES 10, 11, 12. — Section of uniform actual line with distributed leakance, equivalent T and equivalent Π.

$$\rho' = \frac{r'}{2} = r_o \tanh \lambda a \qquad \text{ohms,} \qquad (79)$$

and

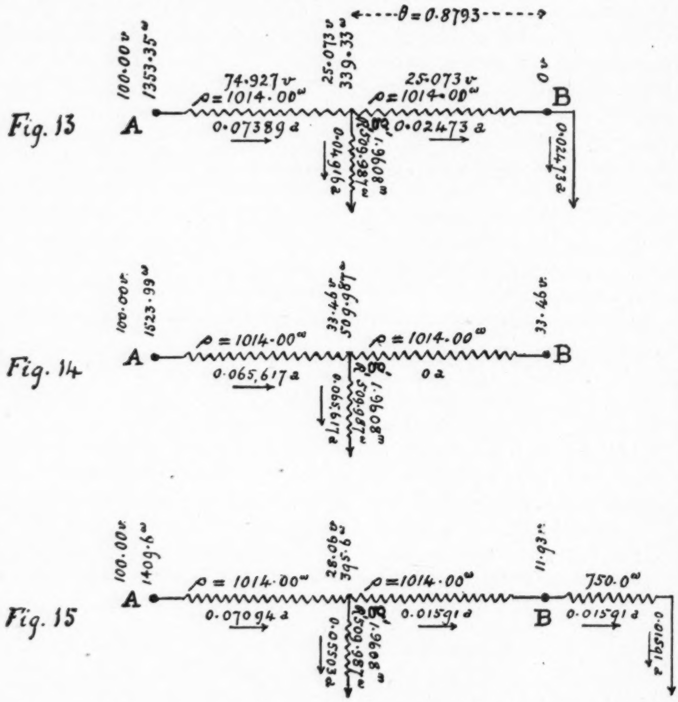
$$g' = \frac{r_o \sinh La}{r_o^2} = \frac{\sinh La}{r_o} \qquad \text{mhos,} \qquad (80)$$

or

$$R' = \frac{r_o^2}{r_o \sinh La} = \frac{r_o}{\sinh La} \qquad \text{ohms.} \qquad (81)$$

That is, the half resistance ρ' is to be equal to the sending-end resistance of each half of the actual line at O, when grounded at A and B; while the resistance of the central leak is to be r_o^2 divided by the receiving-end resistance of the whole line grounded.

Thus, considering the actual smooth line of 500 km. length of which the artificial line represented in Figures 5, 6, and 7 is the external equivalent, we have $r = 5.051$ ohms per km., $g = 2.4499$ micromhos per km., $a = 0.0035172$ hyps. per km., $r_0 = 1436.13$ ohms, $La = 1.7586$ hyps., $\lambda a = 0.8793$ hyp. From (79) we obtain :



FIGURES 13, 14, 15. — Equivalent T of line imitated in Figures 5, 6, and 7. Grounded, freed, and grounded imperfectly.

$$\rho' = 1436.1 \times 0.70607 = 1014.0 \text{ ohms},$$

and by (81) $\quad R' = 1436.1/2.81602 = 509.987$ ohms.

The above values of ρ' and R' have been employed in Figures 13, 14, and 15 to produce a single-section artificial line. It will be seen by comparing these Figures respectively with Figures 5, 6, and 7, that

although the internal distributions of voltage and current differ, the external distributions are identical. That is, the distribution of voltage and current at the ends of, and anywhere external to, the artificial line are identical for the single-section artificial line of Figures 13, 14, and 15; or for the five-section artificial line of Figures 5, 6, and 7; or for the actual smooth uniform line of $a = 0.0035172$ hyp. per km. and $r_0 = 1436.1$ ohms, there imitated.

For brevity and convenience, let a single-section artificial line, like that of Figure 14, formed of a conductor-resistance r' ohms, with a leak of R' ohms at the centre, be called a T, from the graphical resemblance. Then, any real smooth uniform line may be replaced by its equivalent T, without any change in the electrical system external to the T, after the steady state has been attained. This proposition, like the rest, applies not only to a continuous-current system, but also to any single-frequency alternating-current system.

In duplex and multiplex telegraphy, artificial lines are required to balance real lines, not only in the steady state, but also in the preceding unsteady state; so that it is not possible to employ an equivalent T for such artificial lines. In telephony, however, it is commonly believed that the electrical phenomena in ordinary conversation are substantially steady state single-frequency phenomena, and that the conditions in the unsteady state are so transient that they may be practically ignored. If this is correct, then it follows that, except for purposes of adjustment, and of convenience in altering the length of line, there is nothing to be gained by employing a multisection artificial line for embodying the laboratory equivalent of an actual line. In other words, a single-section artificial line of properly selected constants should be just as good as a multisection artificial line, in regard to carrying on conversation. It is important to have this question settled experimentally. The experiment, if unsuccessful, cannot, however, be competent to determine whether the unsteady state enters appreciably into the phenomena of practical telephonic transmission, owing to the presence of multiple frequencies or harmonics.

Conversely, if we have a given T line, we can determine its hyperbolic angle and surge-resistance; that is, we can determine the actual smooth uniform line to which it corresponds; for in Figures 10 and 11

$$\sinh \theta = \sinh \lambda a = \sqrt{\frac{\rho'}{2R'}} \qquad (82)$$

and

$$r_o = \sqrt{\rho'(\rho' + 2R')} \qquad \text{ohms.} (83)$$

Thus, the T of Figures 13, 14, and 15 has $\rho' = 1014$ ohms and $R' = 509.99$ ohms. Hence by (82), $\sinh \theta = 0.99707$, from which the semi-angle $\theta = 0.8793$ hyp., which is also the semi-angle λa of the equivalent smooth line. Again, $r_o = 1436.1$ ohms by (83). These are the constants for the line simulated by the T.

Instead of a T, or conductor with a single central leak, we may substitute for any actual smooth uniform line a conductor with two equal terminal leaks, as shown in Figure 12. Such a conductor may be called a Π for convenience and brevity. In Figure 12, the values to be assigned to the conductor-resistances r'' and leak resistances R'' R'' ohms, in order to replace a smooth line of length L kms., semi-length λ kms., attenuation-constant a hyp. per km., and surge-resistance r_o ohms, are:

$$r'' = r_o \sinh La \qquad \text{ohms,} \quad (84)$$

$$R'' = \frac{r_o^2}{r_o \tanh \lambda a} = \frac{r_o}{\tanh \lambda a} \qquad \text{ohms,} \quad (85)$$

or

$$g'' = \frac{r_o \tanh \lambda a}{r_o^2} = \frac{\tanh \lambda a}{r_o} \qquad \text{mhos.} \quad (86)$$

That is, the conductor resistance r'' must be equal to the receiving-end resistance of the imitated line when grounded, and each leak must be the square of the surge-resistance divided by the sending-end resistance of half the imitated line grounded.

Thus, with $L = 500$ kms., $\lambda = 250$ kms., $a = 0.0035172$ hyp. per km., $r_o = 1436.13$ ohms, $La = 1.7586$ hyps., $\lambda a = 0.8793$ hyp., we have $r'' = 1436.13 \times 2.81602 = 4044.2$ ohms, and $R'' = 1436.13/0.70607 = 2034.05$ ohms. These values have been used in Figures 16, 17, and 18 to construct the Π there indicated. It will be seen by comparing these Figures with 5, 6, 7, and with 10, 11, 12, respectively, that the external distributions of resistance, conductance, voltage, current, and power are the same for all.

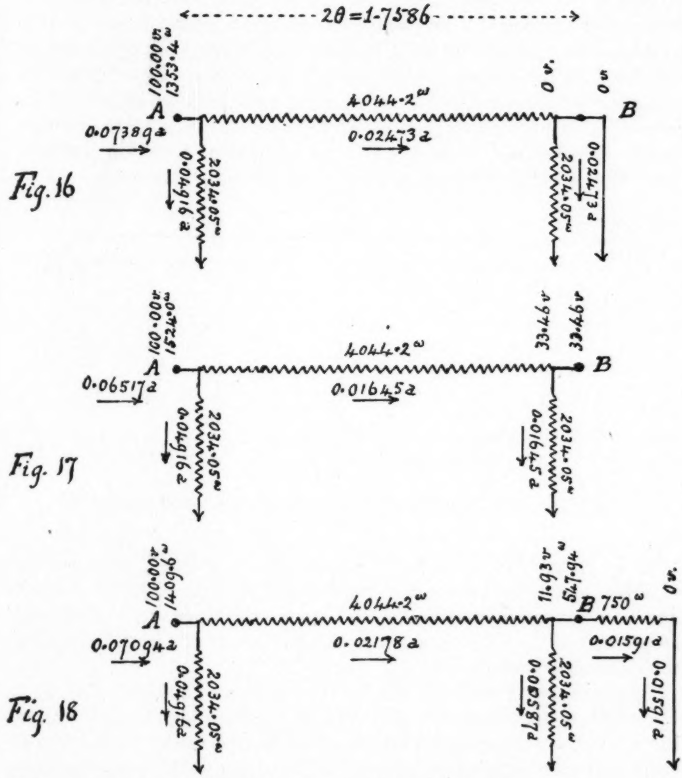
Consequently, any smooth uniform line in the steady state, carrying either continuous or single-frequency alternating currents, may be completely replaced, so far as concerns all external conditions, either by one equivalent T, or by one equivalent Π. Either of these forms of equivalent conductor may be selected for replacing the line, according to convenience.

Conversely, any given Π may have its hyperbolic angle and surge-resistance determined; that is, its equivalent smooth uniform line can be determined by the following formulas:

$$\tanh\theta = \tanh\lambda a = \sqrt{\frac{g''r''}{2+g''r''}} = \sqrt{\frac{r''}{2R''+r''}}, \qquad (87)$$

and

$$r_o \sqrt{\frac{r''}{g''(2+g''r'')}} = R''\sqrt{\frac{r''}{2R''+r''}} = R''\tanh\theta \text{ ohms.} \qquad (88)$$



FIGURES 16, 17, 18. — Equivalent Π of line imitated in Figures 5, 6, and 7. Grounded, freed, and grounded imperfectly.

Thus, with $r'' = 4044.2$ ohms, and $R'' = 2034.05$ ohms, as in Figures 16, 17, and 18, we have $\tanh\theta = \sqrt{4044.2/8112.3} = 0.70606$, and $r_o = 2034.05 \times 0.70606 = 1436.1$ ohms, as before.

It is possible, by known methods of substitution, to derive combinations of resistance and leakance that shall replace a given T or Π; as, for instance, a combination like that shown in Figure 19. All such conductors must manifestly be either graphically symmetrical about a vertical through their centre 0, or must be reducible to such symmetry. In general, these combinations are unnecessarily complex and have little practical interest. From this standpoint, a multiple-section artificial line like that of Figures 5, 6, and 7 may be regarded as a complex substitute for the simple T of Figure 11, or the simple Π of Figure 12.

It may be observed, however, that the total leakage of current to ground in corresponding Figures is the same for a smooth uniform line, its equivalent T, equivalent Π, or equivalent 5-section artificial line. On reflection, this proposition is almost self-evident.

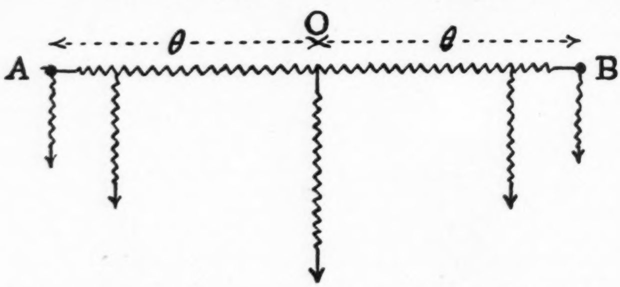


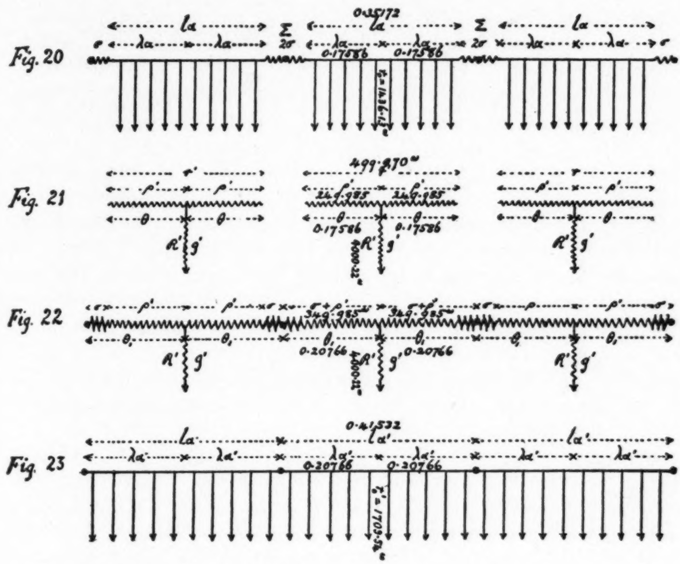
FIGURE 19. — Complex substitute for an actual line of distributed leakance.

As an instance of the use of substituting equivalent T's for sections of actual line, consider the case represented in Figure 20, of a uniform line of attenuation-constant a, and surge-resistance r_0, loaded with resistances of $\Sigma = 2\,\sigma$ ohms, at uniform intervals of l kms. Required the equivalent smooth line.

First substitute uniform T's for the sections of uniform line, as in Figure 21, by formulas (79), (80), and (81). Then load the T's by adding σ to each end, as in Figure 22. Finally replace the loaded T's by their equivalent lengths of smooth line, as in Figure 23, using formulas (82) and (83). We deduce by this process the following results:

$$\sinh \lambda a' = \sinh \lambda a \sqrt{1 + \frac{\sigma \coth \lambda a}{r_0}}, \tag{89}$$

$$\cosh \lambda a' = \cosh \lambda a \sqrt{1 + \frac{\sigma \tanh \lambda a}{r_0}}, \tag{90}$$



FIGURES 20, 21, 22, 23. — Reduction of a uniform actual line with loads in series to an equivalent unloaded actual line.

$$\tanh \lambda a' = \tanh \lambda a \sqrt{\frac{1 + \dfrac{\sigma \coth \lambda a}{r_o}}{1 + \dfrac{\sigma \tanh \lambda a}{r_o}}} = \sqrt{\tanh \lambda a \tanh (\lambda a + \delta)}, \qquad (91)$$

if $\sigma < r_o$ and $\dfrac{\sigma}{r_o} = \tanh \delta$; or

$$\coth \lambda a' = \sqrt{\coth \lambda a \coth (\lambda a + \delta')}, \qquad (92)$$

if $\sigma > r_o$ and $\dfrac{\sigma}{r_o} = \coth \delta'$.

Also $$\sinh la' = \sinh la \sqrt{1 + \frac{2\sigma}{r_o} \coth la + \left(\frac{\sigma}{r_o}\right)^2}; \qquad (93)$$

$$\cosh la' = \cosh la + \frac{\sigma}{r_o} \sinh la; \qquad (93a)$$ [4]

[4] This formula (93a) was first published by Dr. Campbell. See Bibliography.

$$r_o' = r_o\sqrt{\left(\tanh \lambda a + \frac{\sigma}{r_o}\right)\left(\coth \lambda a + \frac{\sigma}{r_o}\right)} \text{ ohms;} \quad (94)$$

$$= r_o\sqrt{1 + \frac{2\sigma}{r_o}\coth la + \left(\frac{\sigma}{r_o}\right)^2} \text{ ohms;} \quad (95)$$

$$\frac{r_o'}{r_o} = \frac{\sinh la'}{\sinh la}. \quad (96)$$

Thus, if a uniform line of attenuation-constant $a = 0.0035172$ hyp./km., and surge-resistance $r_o = 1436.13$ ohms, has a resistance $\Sigma = 200$ ohms, inserted at intervals of 100 kms., required the corresponding constants of the loaded line. Here, as indicated in Figure 21, $\sigma = 100$ ohms and $\lambda a = 0.17586$ hyp. If we compute the equivalent T's of the sections of unloaded line, we find $\rho' = 249.985$ ohms and $R' = 4000.215$ ohms. The hyperbolic corrections for these lengths of sections are thus only 0.015 ohm in conductor-resistance and 0.215 ohm in leak-resistance. Adding on the loads to the ends of the T's, we have, as in Figure 22, $\rho' = 349.985$ ohms and $R' = 4000.215$ ohms. Using formulas (82) and (83), we obtain for the equivalent smooth line $\lambda a' = 0.20766$ hyp., $la' = 0.41532$ hyp., and $r_o' = 1709.54$ ohms. The apparent conductor-resistance of the loaded line is, therefore, $r_o'la' = 710.06$ ohms, or 10.06 ohms more than the actual resistance of conductor and loads. The apparent total leak $r_o'/la' = 4116.2$ ohms, or 116.2 ohms in excess of the actual total leak.

As an example of the use of substituting equivalent Π's for sections of smooth line, consider the case represented in Figure 24 of a uniform line of attenuation-constant a, and surge-resistance r_o, loaded with uniform leakances of Γ mhos at uniform intervals of l kms. Required the constants of the equivalent smooth line.

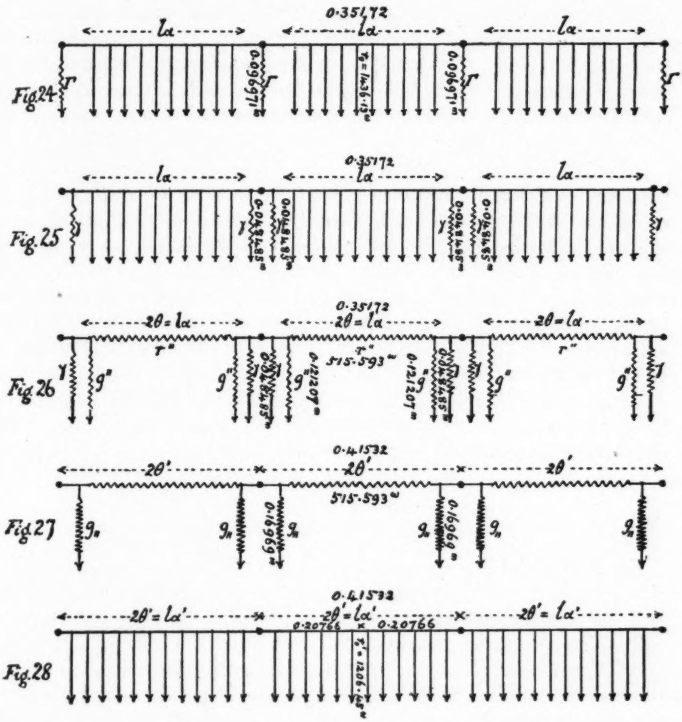
First divide the leakage conductances into equal parts $\gamma = \Gamma/2$, as in Figure 25. Then substitute for the unloaded line sections their equivalent Π's by formulas (84), (85), and (86), as in Figure 26. Next add on the terminal leakances γ to the pillars of the Π, as in Figures 27. Finally, deduce as in Figure 28, by formulas (87) and (88), the equivalent smooth line.

We also obtain by this process the following relations: —

$$\tanh \theta' = \tanh \lambda a' = \tanh \lambda a \sqrt{\frac{1 + \gamma r_o \coth \lambda a}{1 + \gamma r_o \tanh \lambda a}} \quad (97)$$

$$r_o' = \frac{r_o}{\sqrt{(1 + \gamma r_o \tanh \lambda a)(1 + \gamma r_o \coth \lambda a)}} \text{ ohms.} \quad (98)$$

Thus, in Figure 24, the load leaks have resistances of 10,312 ohms, or conductances of 0.096971 millimho, the line sections have lengths $l = 100$ kms., the attenuation-constant 0.0035172 hyp. per km., the hyperbolic angles $la = 0.35172$, $\lambda a = 0.17586$, $r_o = 1436.13$ ohms.



FIGURES 24 TO 28. — Reduction of a uniform actual line with loads in derivation to an equivalent unloaded actual line.

Required the corresponding constants of the loaded line. The load leaks are bisected in Figure 25 to 0.048486 millimho each. The equivalent Π of each unloaded line section, as shown in Figure 26, has a resistance of $r'' = 515.593$ ohms and a leakance g'' of 0.121207 millimho. Adding the γ loads to the pillars of the Π, we have, as in Figure 27, $g_{\prime\prime} = 0.16969$ millimho. Finally, reducing the loaded Π's

to their equivalent smooth-line sections by formulas (87) and (88), we obtain $\lambda a' = 0.20766$ hyp. or $la' = 0.41532$ hyp. and $r_o' = 1206.45$ ohms, as in Figure 28. The apparent conductor resistance of a section of loaded line is $la'r_o' = 501.06$ ohms, or 1.06 ohms in excess of the total actual resistance. The apparent total leakance of a section is 0.34425 millimho, or 0.00272 millimho in excess of the total actual leakance.

It may be observed by comparing Figures 20–23 and 24–28, or formulas (91) and (97), that if loads are applied at assigned uniform distances along a smooth line, a leak load Γ will produce the same equivalent attenuation-constant as a resistance load Σ in the conductor, if $\frac{\sigma}{\gamma} = r_o^2$; that is, if the resistance $1/\gamma$ of a semi-leak be a third proportional to the resistance σ of a semi-conductor-load, and the surge-resistance of the unloaded line. In other words, the attenuation-constant of the loaded line will be the same, whether the loads are inserted in series, or applied in derivation, provided that $\sigma : r_o :: r_o : 1/\gamma$. The surge-resistance of the loaded line will not, however, be the same in these two cases. The surge-resistance will be less with leaks than with series coils. The two values have the unloaded surge-resistance as their geometrical mean.

In all cases of direct-current lines, loads, either in series coils or in leaks, necessarily increase the attenuation-constant of the line. With alternating-current lines, this limitation is removed.

Summary of Conclusions.

Every artificial line composed of similar mid-leak sections, carrying either continuous or alternating currents in the steady state, may be reduced trigonometrically to its equivalent smooth line, and reciprocally. The resistance, current, and voltage at the various junctions and leaks along the line are simple hyperbolic functions of their angles.

Every smooth line in the steady state, carrying either continuous or alternating currents, may be externally completely replaced by one and only one T, or single-section mid-leak artificial line; or by one and only one Π, or single conductor with equal terminal leaks, and reciprocally. This proposition has numerous applications in telegraphy, telephony, power transmission, and distribution.

List of Symbols employed.

a = Attenuation-constant of a smooth line, or of an artificial line after being corrected for lumpiness (hyps. per km.).

a' = The uncorrected attenuation-constant of an artificial line, or the attenuation-constant of a smooth line after being loaded (hyps. per km.).

L = Total length of a line (kms.).

L_2 = A length of line, partial or total, measured from receiving end (kms.).

l = Length of a section of artificial or real line (kms.).

λ = Length of a semi-section of artificial or real line (kms.).

θ, λa = Hyperbolic angles of a semi-section of line (hyps.).

Θ = Total hyperbolic angle from far end (hyps.).

ϕ, ϕ' = Hyperbolic angles of a terminal load (hyps.).

la, La, L_2a = Hyperbolic angles of a section, or length of line (hyps.).

r = Linear conductor-resistance of a line (ohms per km.).

r' = Conductor resistance of a section of artificial line (ohms).

ρ, ρ' = Conductor resistance of a semi-section of artificial line (ohms).

R, R' = Resistance of a central leak in a section of artificial line or T (ohms).

Σ = Resistance of a series load in a line (ohms).

σ = Resistance of a semi-series load or of a single terminal load (ohms).

r'' = Conductor resistance of a Π (ohms).

R'' = Resistance of each leak of a Π (ohms).

r_o' = Nominal or apparent surge-resistance of an artificial line, uncorrected for lumpiness (ohms)

r_o = Surge-resistance of a smooth line, or of an artificial line corrected for lumpiness (ohms).

r_o'', r_o''' = Surge-resistances at receiving ends of terminally loaded lines (ohms).

R_f, R_g = Sending-end resistance of a line respectively freed and grounded at far end (ohms).

$R_{f\sigma}$, $R_{g\sigma}$ = Sending-end resistance of a line grounded at far end through terminal load (ohms).

R'_{nf}, R'_{ng} = Sending-end resistance of a line at nth leak, excluding same (ohms).

R'_{fn}, R'_{gn} = Sending-end resistance of a line at nth leak, including same (ohms).

R_l, $R_{l\sigma}$ = Receiving-end resistance of a line grounded at far end directly, or through terminal load (ohms).

e_m, e_n, e_o = Voltage at mth junction, nth junction and far end (volts).

ϵ_m, ϵ_n = Voltage at mth or nth leak (volts).
ϵ = Base of Napierian logarithms.
I_m, I_n, I_o = Currents in line at sending-end, nth junction, and far end (amperes).
i_m, i_n = Currents in mth and nth leaks (amperes).
g = Linear leakance of smooth line (mhos per km.).
g' = Conductance of central leak in a T or in a section of artificial line (mhos).
g'', $g_{''}$ = Conductance of each leak in a Π (mhos).
m, n = Total number, and reference number, of section junctions in artificial line.
a, a' = Receiving instrument magnetic constants.
n_1 = Number of turns in receiving instrument windings.
F = Force, or torque, exerted by receiving electromagnetic instrument (dynes, or dyne-perp. cms.).
p = Numerical exponent.
Γ, γ = Load leaks, and semi-leaks (mhos).

Appendix.

The demonstrations of the various formulas appearing in the foregoing paper have been omitted in order to save space. Nearly all of these formulas are, however, based upon and derived from the following propositions:

(1) Any alternating continued fraction is expressible as a constant continued fraction. Thus to n stages:

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+}}}} = \frac{b}{\sqrt{ab}} \times \cfrac{1}{\sqrt{ab}+\cfrac{1}{\sqrt{ab}+\cfrac{1}{\sqrt{ab}+\cfrac{1}{\sqrt{ab}+}}}}$$

(2) Any constant continued fraction is expressible as a simple single fraction or ratio of a hyperbolic sine and cosine. Thus the nth convergent of

$$\cfrac{1}{c+\cfrac{1}{c+\cfrac{1}{c}}} = \frac{\sinh n\theta}{\cosh (n+1)\theta} \text{ if } n \text{ is even; or}$$

$$= \frac{\cosh n\theta}{\sinh (n+1)\theta} \quad \text{if } n \text{ is odd,}$$

where $$\theta = \sinh^{-1}\left(\frac{c}{2}\right).$$

(3) Any terminally loaded constant-continued fraction is expressible as a simple fraction or ratio of hyperbolic sine and cosine. Thus the nth ascending convergent of

$$\cfrac{1}{c+\cfrac{1}{c+\cfrac{1}{c+m}}} = \frac{\sinh (n\theta+\phi)}{\cosh [(n+1)\theta+\phi]} \quad \text{if } n \text{ is even; or}$$

$$= \frac{\cosh (n\theta+\phi)}{\sinh [(n+1)\theta+\phi]} \quad \text{if } n \text{ is odd;}$$

where ϕ is an auxiliary hyperbolic angle.

(4) The sending-end resistance of any artificial line composed of similar sections, whether the leaks are in the middle or not, may always be expressed as a terminally loaded alternating continued fraction.

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